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Triple fixed point theorems on FLM algebras

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Abstract

This paper considers tripled fixed point theorems on unital without of order semi-simple fundamental locally multiplicative topological algebras (abbreviated by FLM algebras).

MSC: 46H

Keywords: tripled fixed point; fundamental topological algebras; FLM algebras; holomorphic function; semi-simple algebras; without of order

1 Introduction

Ansari in [1] introduced the notion of fundamental topological spaces and algebras and proved Cohen's factorization theorem for these algebras. A topological linear space $\mathcal{A}$ is said to be fundamental if there exists $b > 1$ such that for every sequence (x_n) of $\mathcal{A}$, the convergence of $b^n(x_n - x_{n-1})$ to zero in $\mathcal{A}$ implies that (x_n) is Cauchy. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

A fundamental topological algebra is called locally multiplicative if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. The fundamental locally multiplicative topological algebras (FLM) were introduced by Ansari in [2]. Some celebrated theorems in Banach algebras were generalized to FLM algebras in [3], and authors investigated some fixed points theorems for holomorphic functions on these algebras (see Theorems 3.5, 3.6 and 3.7 of [3]).

An algebra $\mathcal{A}$ is called without of order if for every $a, b \in \mathcal{A}$, $ab = 0$, then $a = 0$ or $b = 0$.

In [4], Bhaskar and Lakshmikantham introduced the notions of a mixed monotone mapping and a coupled fixed point, proved some coupled fixed point theorems for the mixed monotone mapping and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Also, Samet and Vetro studied a coupled fixed point of N-order in [5]. There are many works on a coupled fixed point of contraction, weak contraction and generalized contraction mappings on various metric spaces such as [6–9].

Let $\mathcal{A}$ be a metric space and let $F : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be a function. An element $(x, y, z) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ is said to be a tripled fixed point of the mapping F if $F(x, y, z) = F(x, z, y) = x$, $F(y, x, z) = F(y, z, x) = y$ and $F(z, x, y) = F(z, y, x) = z$. Tripled fixed point theorems in partially ordered metric spaces were studied by Berinde and Borcut in [10], and this concept was considered by Aydi *et al.* for weak compatible mappings in abstract metric spaces [11].

In this paper, at first (Section 2) we obtain some basic results for FLM algebras, and next we consider tripled fixed point theorems on FLM algebras.

2 Some results on FLM algebras

By $\Omega_{\mathcal{A}}$ we mean the set of all elements $a \in \mathcal{A}$ such that $\rho_{\mathcal{A}}(a) < 1$, where $\rho_{\mathcal{A}}(a)$ is the spectral radius of $a \in \mathcal{A}$. We denote the center of topological algebra $\mathcal{A}$ by $Z(\mathcal{A})$ such that

$$Z(\mathcal{A}) = \{a \in \mathcal{A} : ax = xa \text{ for all } x \in \mathcal{A}\}.$$

Definition 2.1 Let $(\mathcal{A}, d)$ be a metrizable topological algebra. We say $\mathcal{A}$ is a submultiplicatively metrizable topological algebra if

$$d(0, xyz) \leq d(0, x)d(0, y)d(0, z) \quad \text{and} \quad d(0, \lambda x) < |\lambda| d(0, x)$$

for each $x, y, z \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. For abbreviation, we denote $d_{\mathcal{A}}(0, x)$ by $D_{\mathcal{A}}(x)$ for any $x \in \mathcal{A}$.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be metric spaces with meters $d_{\mathcal{A}}$, $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$, respectively. Then $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ becomes a metric space with the following meter:

$$d\big((a_1, b_1, c_1), (a_2, b_2, c_2)\big) = d_{\mathcal{A}}(a_1, a_2) + d_{\mathcal{B}}(b_1, b_2) + d_{\mathcal{C}}(c_1, c_2) \tag{2.1}$$

for every $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$. When $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are algebras, then by the usual point-wise definitions for addition, scalar multiplication and product, $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ becomes an algebra.

Proposition 2.2 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be complete metrizable FLM algebras with submultiplicative meters $d_{\mathcal{A}}$, $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$, respectively. Then $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ is a complete metrizable FLM algebra with a submultiplicative meter d.*

Proof Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be FLM algebras with meters $d_{\mathcal{A}}$, $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$, respectively. By the definition of FLM algebras, obviously, $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ is a complete metrizable FLM algebra with a meter d (the meter defined in (2.1)). For submultiplicativity, we have

$$\begin{aligned}
&d\big((0,0,0), (a_1a_2, b_1b_2, c_1c_2)\big) \\
&\quad = d_{\mathcal{A}}(0, a_1a_2) + d_{\mathcal{B}}(0, b_1b_2) + d_{\mathcal{C}}(0, c_1c_2) \\
&\quad \leq d_{\mathcal{A}}(0, a_1)d_{\mathcal{A}}(0, a_2) + d_{\mathcal{B}}(0, b_1)d_{\mathcal{B}}(0, b_2) + d_{\mathcal{C}}(0, c_1)d_{\mathcal{C}}(0, c_2) \\
&\quad \leq d_{\mathcal{A}}(0, a_1)d_{\mathcal{A}}(0, a_2) + d_{\mathcal{A}}(0, a_1)d_{\mathcal{B}}(0, b_2) + d_{\mathcal{A}}(0, a_1)d_{\mathcal{C}}(0, c_2) \\
&\qquad + d_{\mathcal{B}}(0, b_1)d_{\mathcal{A}}(0, a_2) + d_{\mathcal{B}}(0, b_1)d_{\mathcal{B}}(0, b_2) \\
&\qquad + d_{\mathcal{B}}(0, b_1)d_{\mathcal{C}}(0, c_2) + d_{\mathcal{C}}(0, c_1)d_{\mathcal{A}}(0, a_2) \\
&\qquad + d_{\mathcal{C}}(0, c_1)d_{\mathcal{B}}(0, b_2) + d_{\mathcal{C}}(0, c_1)d_{\mathcal{C}}(0, c_2) \\
&\quad = d\big((0,0,0), (a_1, b_1, c_1)\big)d\big((0,0,0), (a_2, b_2, c_2)\big)
\end{aligned} \tag{2.2}$$

for every $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$. Also,

$$\begin{aligned}
d\big((0,0,0), (\lambda a, \lambda b, \lambda c)\big) &= d_{\mathcal{A}}(0, \lambda a) + d_{\mathcal{B}}(0, \lambda b) + d_{\mathcal{C}}(0, \lambda c) \\
&< |\lambda| d_{\mathcal{A}}(0, a) + |\lambda| d_{\mathcal{B}}(0, b) + |\lambda| d_{\mathcal{C}}(0, c)
\end{aligned}$$

$$= |\lambda|\big(d_{\mathcal{A}}(0,a) + d_{\mathcal{B}}(0,b) + d_{\mathcal{C}}(0,c)\big)$$
$$= |\lambda|\big(d\big((0,0,0),(a,b,c)\big)\big). \tag{2.3}$$

Therefore, (2.2) and (2.3) show that d is submultiplicative. □

Similar to Definition 2.1, we write $D_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}(a,b,c)$ as an abbreviation for $d((0,0,0),(a,b,c))$. We recall the following theorem from [3].

Theorem 2.3 [3, Theorem 3.3] *Let $\mathcal{A}$ be a complete metrizable FLM algebra with a submultiplicative meter $d_{\mathcal{A}}$. Then $\rho(x) = \lim_{n\to\infty} D_{\mathcal{A}}(x^n)^{1/n}$.*

Lemma 2.4 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be complete metrizable FLM algebras with submultiplicative meters $d_{\mathcal{A}}$, $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$, respectively. Then*

$$\rho(x,y,z) \le \rho_{\mathcal{A}}(x) + \rho_{\mathcal{B}}(y) + \rho_{\mathcal{C}}(z)$$

for any element $(x,y,z) \in \mathcal{A}\times\mathcal{B}\times\mathcal{C}$.

Proof For given $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$, we have $\rho_{\mathcal{A}}(a) = \lim_{n\to\infty} D_{\mathcal{A}}(a^n)^{1/n}$, $\rho_{\mathcal{B}}(b) = \lim_{n\to\infty} D_{\mathcal{B}}(b^n)^{1/n}$ and $\rho_{\mathcal{C}}(c) = \lim_{n\to\infty} D_{\mathcal{C}}(c^n)^{1/n}$ (Theorem 2.3). From Proposition 2.2, it follows that $\mathcal{A}\times\mathcal{B}\times\mathcal{C}$ is a complete metrizable FLM algebra with a submultiplicative meter d. Then again, Theorem 2.3 implies that

$$\begin{aligned}
\rho(x,y,z) &= \lim_{n\to\infty} D_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}\big((x,y,z)^n\big)^{\frac{1}{n}} = \lim_{n\to\infty} D_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}\big(\big(x^n,y^n,z^n\big)\big)^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\big(D_{\mathcal{A}}\big(x^n\big) + D_{\mathcal{B}}\big(y^n\big) + D_{\mathcal{C}}\big(z^n\big)\big)^{\frac{1}{n}} \\
&\le \lim_{n\to\infty} D_{\mathcal{A}}\big(x^n\big)^{\frac{1}{n}} + \lim_{n\to\infty} D_{\mathcal{B}}\big(y^n\big)^{\frac{1}{n}} + \lim_{n\to\infty} D_{\mathcal{C}}\big(z^n\big)^{\frac{1}{n}} \\
&= \rho_{\mathcal{A}}(x) + \rho_{\mathcal{B}}(y) + \rho_{\mathcal{C}}(z)
\end{aligned} \tag{2.4}$$

for every $x \in \mathcal{A}$, $y \in \mathcal{B}$ and $z \in \mathcal{C}$. □

Similar to $\Omega_{\mathcal{A}}$ and $Z(\mathcal{A})$, we define these sets for $\mathcal{A}\times\mathcal{A}\times\mathcal{A}$ as follows:

$$\Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} = \big\{(x,y,z) \in \mathcal{A}\times\mathcal{A}\times\mathcal{A} : \rho(x,y,z) < 1\big\},$$

and

$$\begin{aligned}
Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A}) &= \big\{(x,y,z) \in \mathcal{A}\times\mathcal{A}\times\mathcal{A} : (x,y,z)(a,b,c) = (a,b,c)(x,y,z), \\
&\qquad \text{for every } a,b,c \in \mathcal{A}\big\} \\
&= \big\{(x,y,z) \in \mathcal{A}\times\mathcal{A}\times\mathcal{A} : (xa,yb,zc) = (ax,by,cz), \\
&\qquad \text{for every } a,b,c \in \mathcal{A}\big\}.
\end{aligned}$$

Clearly, if $(x,y,z) \in Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$, then $x,y,z \in Z(\mathcal{A})$ and $Z(\mathcal{A}) \subseteq Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$. Also, if $(x,y,z) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}}$, then $(x,0,0)$, $(0,y,0)$ and $(0,0,z)$ are in $\Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}}$, and by Lemma 2.4 and its proof, we have $x,y,z \in \Omega_{\mathcal{A}}$.

Let $E(\mathcal{A})$ be the set of all elements $x \in \mathcal{A}$ for which $E(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ can be defined. If $\mathcal{A}$ is a complete metrizable FLM algebra, then $E(\mathcal{A}) = \mathcal{A}$ ([12, Theorem 5.4]). Therefore, in the light of Theorem 5.4 of [12] and Proposition 2.2, we have the following theorem.

Theorem 2.5 *Let $\mathcal{A}$ be a complete metrizable FLM algebra, then $E(\mathcal{A} \times \mathcal{A} \times \mathcal{A}) = \mathcal{A} \times \mathcal{A} \times \mathcal{A}$.*

3 Tripled fixed point theorems

In this section, we consider some results about tripled fixed point theorems on unital complete semi-simple metrizable FLM algebras, and we extend these results to Banach algebras. By $\mathrm{id}_{\mathcal{A}}$, we mean the identity map on $\mathcal{A}$.

Theorem 3.1 *Let $\mathcal{A}$ be a unital without of order complete semi-simple metrizable FLM algebra with a submultiplicative meter $d_{\mathcal{A}}$. If $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \subseteq \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow \Omega_{\mathcal{A}}$ is a holomorphic map that satisfies the conditions $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i\,\partial y^j\,\partial z^k}(0,0,0) = 0$, where $i+j+k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i\,\partial y^j\,\partial z^k}(0,0,0) = 0$, where $i+j+k = 3$, $i,j,k = 0,1,2,3$, then every $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A} \times \mathcal{A} \times \mathcal{A})$ is a tripled fixed point for F.*

Proof Fix $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A} \times \mathcal{A} \times \mathcal{A})$ and consider the map $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \longrightarrow \Omega_{\mathcal{A}}$ with $f(\alpha,\beta,\gamma) = F(\alpha a, \beta b, \gamma c)$. Clearly, f is a holomorphic function on

$$\left\{(\alpha,\beta,\gamma) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} : \frac{|\theta|}{3} < \frac{1}{\rho(a,b,c)}, |\theta| = \min\{|\alpha|, |\beta|, |\gamma|\},\right.$$
$$\left.\rho_{\mathcal{A}}(a) < \frac{1}{|\alpha|}, \rho_{\mathcal{B}}(b) < \frac{1}{|\beta|}, \rho_{\mathcal{C}}(c) < \frac{1}{|\gamma|}\right\}.$$

Since $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i\,\partial y^j\,\partial z^k}(0,0,0) = 0$, where $i+j+k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i\,\partial y^j\,\partial z^k}(0,0,0) = 0$, where $i+j+k = 3$, $i,j,k = 0,1,2,3$, then F has a Taylor expansion about $(0,0,0)$:

$$\begin{aligned}
F(x,y,z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^i y^j z^k}{i!j!k!}\left(\frac{\partial^{i+j+k} F}{\partial x^i\,\partial y^j\,\partial z^k}\right)(0,0,0) \\
&= x + \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{j} \binom{k-j}{i} x^i y^j z^{k-i-j} \left(\frac{\partial^k F}{\partial x^i\,\partial y^j\,\partial z^{k-i-j}}\right)(0,0,0)
\end{aligned}$$

for every $(x,y,z) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A} \times \mathcal{A} \times \mathcal{A})$. Therefore,

$$\begin{aligned}
F(\alpha a, \beta b, \gamma c) = \alpha a &+ \sum_{k=4}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{j} \binom{k-j}{i} \alpha^i a^i \beta^j b^j \gamma^{k-i-j} c^{k-i-j} \\
&\times \left(\frac{\partial^k F}{\partial x^i\,\partial y^j\,\partial z^{k-i-j}}\right)(0,0,0).
\end{aligned} \tag{3.1}$$

We claim that

$$\sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{j} \binom{k-j}{i} \alpha^i a^i \beta^j b^j \gamma^{k-i-j} c^{k-i-j} \left(\frac{\partial^k F}{\partial x^i\,\partial y^j\,\partial z^{k-i-j}}\right)(0,0,0), \tag{3.2}$$

is zero for every $k \geq 4$. Assume towards a contradiction that there exists $k \geq 4$ such that (3.2) is non-zero. Let $l \geq 4$ be an integer such that

$$\sum_{j=0}^{k} \binom{l}{j} \sum_{i=0}^{j} \binom{l-j}{i} \alpha^i a^i \beta^j b^j \gamma^{l-i-j} c^{l-i-j} \left(\frac{\partial^k F}{\partial x^i \, \partial y^j \, \partial z^{l-i-j}} \right)(0,0,0) \neq 0. \tag{3.3}$$

Suppose that q is an element of $\mathcal{A}$ such that $\rho_{\mathcal{A}}(q) = 0$. Now, we consider the following five cases:

(1) $i = l, j = 0$,
(2) $i = 0, j = l$,
(3) $i + j = l$,
(4) $1 \leq i + j < l$,
(5) $i = j = 0$.

Case (1). In this case, we have $\alpha^l a^l \frac{\partial^l F}{\partial x^l}(0,0,0) \neq 0$. Let $n \geq 1$, by (3.1) and (3.3), we have

$$\begin{aligned} F\left(n^{\frac{1}{l}}\alpha a + n\alpha^l q, \beta b, \gamma c\right) &= n^{\frac{1}{l}}\alpha a + n\alpha^l q + \frac{1}{l!}\left(n^{\frac{1}{l}}\alpha a + n\alpha^l q\right)^l \frac{\partial^l F}{\partial x^l}(0,0,0) \\ &= n^{\frac{1}{l}}\alpha a + n\alpha^l q + \frac{1}{l!}\left(n^l \alpha^{l^2} q^l + l n^{\frac{1}{l}}\alpha a n^{l-1}\alpha^{l(l-1)} q^{l-1} \right. \\ &\quad \left. + \cdots + n\alpha^l a^l\right) \frac{\partial^l F}{\partial x^l}(0,0,0) \\ &= n^{\frac{1}{l}}\alpha a + n\alpha^l \left(q + \frac{1}{l!} a^l \frac{\partial^l F}{\partial x^l}(0,0,0) \right) + P(\alpha) \frac{\partial^l F}{\partial x^l}(0,0,0). \end{aligned} \tag{3.4}$$

In (3.4), by $P(\alpha)$, we mean the remaining part of $(n^{\frac{1}{l}}\alpha a + n\alpha^k q)^k$. Since $a \in Z(\mathcal{A})$, therefore $aq = qa$. Then Lemma 2.4 and Lemma 3.6 of [3] imply

$$\begin{aligned} \rho\left(n^{\frac{1}{l}}\alpha a + n\alpha^l q, \beta b, \gamma c\right) &\leq \rho_{\mathcal{A}}\left(n^{\frac{1}{l}}\alpha a + n\alpha^l q\right) + \rho_{\mathcal{A}}(\beta b) + \rho_{\mathcal{A}}(\gamma c) \\ &< n^{\frac{1}{l}}|\alpha|\rho_{\mathcal{A}}(a) + |\beta|\rho_{\mathcal{A}}(b) + |\gamma|\rho_{\mathcal{A}}(c) \\ &< \mu\left(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b) + \rho_{\mathcal{A}}(c)\right), \end{aligned}$$

where $\mu = \max\{n^{\frac{1}{l}}|\alpha|, |\beta|, |\gamma|\}$. Now, we define a holomorphic function H from $\{\alpha \in \mathbb{C} : 0 < |\alpha| < \frac{1}{\rho(a,b,c)}\}$ into $\mathcal{A}$ as follows:

$$H(\alpha) = \frac{F(n^{\frac{1}{l}}\alpha a + n\alpha^l q, \beta b, \gamma c) - n^{\frac{1}{l}}\alpha a}{n\alpha^l}.$$

By (3.4) we conclude that $H(0) = q + \frac{1}{l!} a^l \frac{\partial^l F}{\partial x^l}(0,0,0)$. Vesentini's theorem ([13, Theorem 3.4.7]) implies that $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\alpha \in \mathbb{C} : 0 < |\alpha| < \frac{1}{\rho(a,b,c)}\}$. Moreover, by the maximum principle, we can write $\rho_{\mathcal{A}}(H(0)) \leq \max_{|\alpha|=1} \rho_{\mathcal{A}}(H(\alpha))$. Then Lemma 3.6 of [3] implies that

$$\begin{aligned} \rho_{\mathcal{A}}\left(q + \frac{1}{l!} a^l \frac{\partial^l F}{\partial x^l}(0,0,0) \right) &\leq \max_{|\alpha|=1} \rho\left(H(\alpha)\right) < \frac{1}{nl!}\rho_{\mathcal{A}}\left(a^l\right)\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^l}(0,0,0) \right) \\ &< \frac{1}{nl!|\alpha|^l}\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^l}(0,0,0) \right). \end{aligned} \tag{3.5}$$

The above inequality holds for every $n \geq 1$. Therefore, if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}\left(q + \frac{1}{l!}a^l \frac{\partial^l F}{\partial x^l}(0,0,0)\right) = 0$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, Theorem 3.4 of [3] implies that $a^l \frac{\partial^l F}{\partial x^l}(0,0,0)$ is in radical of $\mathcal{A}$. Since $\mathcal{A}$ is semi-simple, therefore $a^l \frac{\partial^l F}{\partial x^l}(0,0,0) = 0$. Since $a \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$, so $a^l \neq 0$, and since $\mathcal{A}$ is without of order, therefore $\frac{\partial^l F}{\partial x^l}(0,0,0) = 0$, a contradiction. Thus, our claim is true, and from (3.1), we conclude that $F(a,b,c) = a$. Similarly, we have $F(a,c,b) = a$, $F(b,a,c) = F(b,c,a) = b$ and $F(c,a,b) = F(c,b,a) = c$.

Case (2). In this case, we have $\beta^l b^l \frac{\partial^l F}{\partial y^l}(0,0,0) \neq 0$. Again, by (3.1) and (3.3), we have

$$\begin{aligned} F\left(\alpha a + n\beta^l q, n^{\frac{1}{l}}\beta b, \gamma c\right) &= \alpha a + n\beta^l q + \frac{1}{l!} n\beta^l b^l \frac{\partial^l F}{\partial y^l}(0,0,0) \\ &= \alpha a + n\beta^l\left(q + \frac{1}{l!}b^l \frac{\partial^l F}{\partial y^l}(0,0,0)\right). \end{aligned}$$

Again, by Lemma 2.4 and Lemma 3.6 of [3], we have

$$\begin{aligned} \rho\left(\alpha a + n\beta^l q, n^{\frac{1}{l}}\beta b, \gamma c\right) &\leq \rho_{\mathcal{A}}\left(\alpha a + n\beta^l q\right) + \rho_{\mathcal{A}}\left(n^{\frac{1}{l}}\beta b\right) + \rho_{\mathcal{A}}(\gamma c) \\ &< |\alpha|\rho_{\mathcal{A}}(a) + n^{\frac{1}{l}}|\beta|\rho_{\mathcal{A}}(b) + |\gamma|\rho_{\mathcal{A}}(c) \\ &< \mu\left(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b) + \rho_{\mathcal{A}}(c)\right), \end{aligned}$$

where $\mu = \max\{|\alpha|, n^{\frac{1}{l}}|\beta|, |\gamma|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, n^{\frac{1}{l}}|\beta|, |\gamma|\}\}$ into $\mathcal{A}$ as follows:

$$H(\alpha) = \frac{F(\alpha a + n\beta^l q, n^{\frac{1}{l}}\beta b, \gamma c) - \alpha a}{n\beta^l}.$$

Then from (3.7) it follows that $H(0) = q + \frac{1}{l!}b^l \frac{\partial^l F}{\partial y^l}(0,0,0)$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, n^{\frac{1}{l}}|\beta|, |\gamma|\}\}$. Moreover, Lemma 3.6 of [3] implies that

$$\begin{aligned} \rho_{\mathcal{A}}\left(q + \frac{1}{l!}b^l \frac{\partial^l F}{\partial y^l}(0,0,0)\right) &\leq \max_{|\alpha|=1} \rho\left(H(\alpha)\right) \\ &< \frac{1}{nl!}\rho_{\mathcal{A}}\left(b^l\right)\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial y^l}(0,0,0)\right) \\ &< \frac{1}{nl!|\beta|^l}\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial y^l}(0,0,0)\right). \end{aligned} \tag{3.6}$$

The above inequality holds for every $n \geq 1$. Therefore, if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}\left(q + \frac{1}{l!}b^l \frac{\partial^l F}{\partial y^l}(0,0,0)\right) = 0$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, Theorem 3.4 of [3] implies that $b^l \frac{\partial^l F}{\partial y^l}(0,0,0)$ is in radical of $\mathcal{A}$. Since $\mathcal{A}$ is semi-simple, therefore $b^l \frac{\partial^l F}{\partial y^l}(0,0,0) = 0$. Since $b \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$, so $b^l \neq 0$. By using that $\mathcal{A}$ is without of order, we conclude that $\frac{\partial^l F}{\partial y^l}(0,0,0) = 0$, a contradiction. Thus, our claim is true, and from (3.1), we conclude that $F(a,b,c) = a$. Similarly, we have $F(a,c,b) = a$, $F(b,a,c) = F(b,c,a) = b$ and $F(c,a,b) = F(c,b,a) = c$.

Case (3). In this case, we suppose that $i + j = l$, $i,j \in \{0,1,2,3,\ldots\}$ (without loss of generality, we prove this case for only one i and one j such that $i + j = l$). Again by (3.1) and (3.3), we have

$$\begin{aligned}
&F\bigl(\alpha a + n\alpha^i q, n^{\frac{1}{l-i}}\beta b, \gamma c\bigr) \\
&\quad = \alpha a + n\alpha^i q + \frac{1}{(l-i)!i!}\bigl(\alpha a + n\alpha^i q\bigr)^i n\beta^{l-i} b^{l-i} \frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0) \\
&\quad = \alpha a + n\alpha^i q + \frac{1}{(l-i)!i!}\bigl(n^{i+1}\alpha^{i^2} q^i \beta^{l-i} b^{l-i} \\
&\qquad + i n^i \alpha^{i(i-1)+1} q^{i-1}\beta^{l-i} b^{l-i} a + \cdots + n\alpha^i a^i \beta^{l-i} b^{l-i}\bigr) \frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0) \\
&\quad = \alpha a + n\alpha^i \left(q + \frac{1}{(l-i)!i!} a^i \beta^{l-i} b^{l-i} \frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)\right) \\
&\qquad + P(\alpha)\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0).
\end{aligned} \tag{3.7}$$

By Lemma 2.4 and Lemma 3.6 of [3], we have

$$\begin{aligned}
\rho\bigl(\alpha a + n\alpha^i q, n^{\frac{1}{l-i}}\beta b, \gamma c\bigr) &\leq \rho_{\mathcal{A}}\bigl(\alpha a + n\alpha^i q\bigr) + \rho_{\mathcal{A}}\bigl(n^{\frac{1}{l-i}}\beta b\bigr) + \rho_{\mathcal{A}}(\gamma c) \\
&< |\alpha|\rho_{\mathcal{A}}(a) + n^{\frac{1}{l-i}}|\beta|\rho_{\mathcal{A}}(b) + |\gamma|\rho_{\mathcal{A}}(c) \\
&< \mu\bigl(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b) + \rho_{\mathcal{A}}(c)\bigr),
\end{aligned}$$

where $\mu = \max\{|\alpha|, n^{\frac{1}{l-i}}|\beta|, |\gamma|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, n^{\frac{1}{l-i}}|\beta|, |\gamma|\}\}$ into $\mathcal{A}$ as follows:

$$H(\lambda) = \frac{F(\alpha a + n\alpha^i q, n^{\frac{1}{l-i}}\beta b, \gamma c) - \alpha a}{n\alpha^i}.$$

Then from (3.7) it follows that $H(0) = q + \frac{1}{(l-i)!i!} a^i \beta^{l-i} b^{l-i} \frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, n^{\frac{1}{l-i}}|\beta|, |\gamma|\}\}$. Moreover, Lemma 3.6 of [3] implies that

$$\begin{aligned}
&\rho_{\mathcal{A}}\left(q + \frac{1}{(l-i)!i!} a^i \beta^{l-i} b^{l-i} \frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)\right) \\
&\quad \leq \max_{|\alpha|=1} \rho\bigl(H(\alpha)\bigr) \\
&\quad < \frac{|\beta|^{l-i}}{n(l-i)!i!}\rho_{\mathcal{A}}\bigl(a^i\bigr)\rho_{\mathcal{A}}\bigl(b^{l-i}\bigr)\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)\right) \\
&\quad < \frac{1}{n(l-i)!i!|\alpha|^i}\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)\right).
\end{aligned} \tag{3.8}$$

The above inequality holds for every $n \geq 1$. Therefore, if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}\left(q + \frac{1}{(l-i)!i!}a^i\beta^{l-i}b^{l-i}\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)\right) = 0$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, $a^i\beta^{l-i}b^{l-i}\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0)$ is in radical of $\mathcal{A}$, therefore $a^i\beta^{l-i}b^{l-i}\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0) = 0$. Since $\beta^{l-i} \neq 0$ and $a, b \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$, so $a^i \neq 0$ and $b^{l-i} \neq 0$. Again, by using that $\mathcal{A}$ is without of order, we conclude that $\frac{\partial^l F}{\partial x^i\,\partial y^{l-i}}(0,0,0) = 0$, a contradiction. Thus, our claim is true, and from (3.1), we conclude that $F(a,b,c) = a$. Similarly, we have $F(a,c,b) = a$, $F(b,a,c) = F(b,c,a) = b$ and $F(c,a,b) = F(c,b,a) = c$.

Case (4). Let $1 \leq i+j \leq l$. Then we have $\alpha^i a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0) \neq 0$. Again, by (3.1) and (3.3), we have

$$\begin{aligned}
&F\bigl(\alpha a + n\alpha^i q, \beta b, n^{\frac{1}{l-i-j}}\gamma c\bigr) \\
&\quad = \alpha a + n\alpha^i q + \frac{1}{i!j!(l-i-j)!}\left(\bigl(\alpha a + n\alpha^i q\bigr)^i(\beta b)^j\bigl(n^{\frac{1}{l-i-j}}\gamma c\bigr)^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right) \\
&\quad = \alpha a + n\alpha^i q + \frac{1}{i!j!(l-i-j)!}\bigl(n^{i+1}\alpha^{i^2}q^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j} + ian^i\alpha^{i(i-1)i+1}q^{i-1}\beta^j b^j\gamma^{l-i-j}c^{l-i-j} \\
&\qquad + \cdots + n\alpha^i a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\bigr)\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0) \\
&\quad = \alpha a + n\alpha^i\left(q + \frac{1}{i!j!(l-i-j)!}a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right) \\
&\qquad + P(\alpha)\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0).
\end{aligned} \tag{3.9}$$

Then

$$\begin{aligned}
\rho\bigl(\alpha a + n\alpha^i q, \beta b, n^{\frac{1}{l-i-j}}\gamma c\bigr) &\leq \rho_{\mathcal{A}}\bigl(\alpha a + n\alpha^i q\bigr) + \rho_{\mathcal{A}}(\beta b) + \rho_{\mathcal{A}}\bigl(n^{\frac{1}{l-i-j}}\gamma c\bigr) \\
&< |\alpha|\rho_{\mathcal{A}}(a) + |\beta|\rho_{\mathcal{A}}(b) + n^{\frac{1}{l-i-j}}|\gamma|\rho_{\mathcal{A}}(c) \\
&< \mu\bigl(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b) + \rho_{\mathcal{A}}(c)\bigr),
\end{aligned}$$

where $\mu = \max\{|\alpha|, |\beta|, n^{\frac{1}{l-i-j}}|\gamma|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, |\beta|, n^{\frac{1}{l-i-j}}|\gamma|\}\}$ into $\mathcal{A}$ as follows:

$$H(\alpha) = \frac{F(\alpha a + n\alpha^i q, \beta b, n^{\frac{1}{l-i-j}}\gamma c) - \alpha a}{n\alpha^i}.$$

Then from (3.9) it follows that $H(0) = q + \frac{1}{i!j!(l-i-j)!}(a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0))$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, |\beta|, n^{\frac{1}{l-i-j}}|\gamma|\}\}$. Therefore,

$$\begin{aligned}
&\rho_{\mathcal{A}}\left(q + \frac{1}{i!j!(l-i-j)!}a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right) \\
&\quad \leq \max_{|\alpha|=1}\rho\bigl(H(\alpha)\bigr)
\end{aligned}$$

$$\begin{aligned}
&< \frac{|\beta|^j|\gamma|^{l-i-j}}{ni!j!(l-i-j)!}\rho_{\mathcal{A}}(a^i)\rho_{\mathcal{A}}(b^j)\rho_{\mathcal{A}}(c^{l-i-j})\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right)\\
&< \frac{1}{ni!j!(l-i-j)!|\alpha|^i}\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right). \qquad (3.10)
\end{aligned}$$

Therefore, if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}\left(q + \frac{1}{i!j!(l-i-j)!}a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)\right) = 0$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, $a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0)$ is in radical of $\mathcal{A}$. Since $\mathcal{A}$ is semi-simple, therefore $a^i\beta^j b^j\gamma^{l-i-j}c^{l-i-j}\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0) = 0$. Since $\beta^j \neq 0$, $\gamma^{l-i-j} \neq 0$ and $a, b \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$, so $a^i \neq 0$, b^j and $c^{l-i-j} \neq 0$, we conclude that $\frac{\partial^l F}{\partial x^i\,\partial y^j\,\partial z^{l-i-j}}(0,0,0) = 0$, a contradiction. Thus, our claim is true, and from (3.1), we conclude that $F(a,b,c) = a$. Similarly, we have $F(a,c,b) = a$, $F(b,a,c) = F(b,c,a) = b$ and $F(c,a,b) = F(c,b,a) = c$.

Case (5). Now, let $i = j = 0$. Then we have $\gamma^l c^l \frac{\partial^l F}{\partial z^l}(0,0,0) \neq 0$. Similar to the previous cases, we have

$$\begin{aligned}
F\left(\alpha a + n\gamma^l q, \beta b, n^{\frac{1}{l}}\gamma c\right) &= \alpha a + n\gamma^l q + \frac{1}{l!}n\gamma^l c^l\frac{\partial^l F}{\partial z^l}(0,0,0)\\
&= \alpha a + n\gamma^l\left(q + \frac{1}{l!}c^l\frac{\partial^l F}{\partial z^l}(0,0,0)\right). \qquad (3.11)
\end{aligned}$$

Then

$$\begin{aligned}
\rho\left(\alpha a + n\gamma^l q, \beta b, n^{\frac{1}{l}}\gamma c\right) &\leq \rho_{\mathcal{A}}\left(\alpha a + n\gamma^l q\right) + \rho_{\mathcal{A}}(\beta b) + \rho_{\mathcal{A}}\left(n^{\frac{1}{l}}\gamma c\right)\\
&< |\alpha|\rho_{\mathcal{A}}(a) + |\beta|\rho_{\mathcal{A}}(b) + n^{\frac{1}{l}}|\gamma|\rho_{\mathcal{A}}(c)\\
&< \mu\left(\rho_{\mathcal{A}}(a) + \rho_{\mathcal{A}}(b) + \rho_{\mathcal{A}}(c)\right),
\end{aligned}$$

where $\mu = \max\{|\alpha|, |\beta|, n^{\frac{1}{l}}|\gamma|\}$. Now, we define a holomorphic function H from $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, |\beta|, n^{\frac{1}{l}}|\gamma|\}\}$ into $\mathcal{A}$ as follows:

$$H(\alpha) = \frac{F(\alpha a + n\gamma^l q, \beta b, n^{\frac{1}{l}}\gamma c) - \alpha a}{n\gamma^l}.$$

Then from (3.11) it follows that $H(0) = q + \frac{1}{l!}c^l\frac{\partial^l F}{\partial z^l}(0,0,0)$. Then $\rho_{\mathcal{A}} \circ H$ is a subharmonic function on $\{\eta \in \mathbb{C} : \mu < \frac{1}{\rho(a,b,c)}, \mu = |\eta| = \max\{|\alpha|, |\beta|, n^{\frac{1}{l}}|\gamma|\}\}$, and

$$\begin{aligned}
\rho_{\mathcal{A}}\left(q + \frac{1}{l!}c^l\frac{\partial^l F}{\partial z^l}(0,0,0)\right) &\leq \max_{|\alpha|=1}\rho\left(H(\alpha)\right)\\
&< \frac{1}{nl!}\rho_{\mathcal{A}}(c^l)\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial z^l}(0,0,0)\right)\\
&< \frac{1}{nl!|\gamma|^l}\rho_{\mathcal{A}}\left(\frac{\partial^l F}{\partial z^l}(0,0,0)\right). \qquad (3.12)
\end{aligned}$$

Therefore, if $n \longrightarrow \infty$, then

$$\rho_{\mathcal{A}}\left(q + \frac{1}{l!}c^l \frac{\partial^l F}{\partial z^l}(0,0,0)\right) = 0$$

for every $q \in \mathcal{A}$ with $\rho_{\mathcal{A}}(q) = 0$. Hence, $c^l \frac{\partial^l F}{\partial z^l}(0,0,0)$ is in radical of $\mathcal{A}$. Therefore, $c^l \frac{\partial^l F}{\partial z^l}(0,0,0) = 0$. Since $c \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$, so $c^l \neq 0$, then $\frac{\partial^l F}{\partial z^l}(0,0,0) = 0$, a contradiction. Thus, (3.1) implies that our claim is true, and from (3.1), we conclude that $F(a,b,c) = a$. Similarly, we have $F(a,c,b) = a$, $F(b,a,c) = F(b,c,a) = b$ and $F(c,a,b) = F(c,b,a) = c$.

By gathering the above five cases, we conclude (a,b,c) is a tripled fixed point for F, and since (a,b,c) was arbitrary, so every point of $\Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$ is a tripled fixed point for F. □

Corollary 3.2 *Let $\mathcal{A}$ be a unital without of order semi-simple Banach algebra. If $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \subseteq \mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \Omega_{\mathcal{A}}$ is a holomorphic map that satisfies the conditions $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 3$, $i,j,k = 0,1,2,3$, then every $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \cap Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$ is a tripled fixed point for F.*

In the following theorem, we characterize tripled fixed points of holomorphic functions on FLM algebras.

Theorem 3.3 *Let $\mathcal{A}$ be a unital without of order complete semi-simple metrizable FLM algebra. For given $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \backslash Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$, there is a holomorphic map $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ satisfying the conditions $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 3$, $i,j,k = 0,1,2,3$, such that $F(a,b,c) \neq a$, $F(b,a,c) \neq b$ and $F(c,a,b) \neq c$.*

Proof Let $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \backslash Z(\mathcal{A}\times\mathcal{A}\times\mathcal{A})$. Then there exist $(u,u,u) \in \mathcal{A}\times\mathcal{A}\times\mathcal{A}$ such that

$$(ua, ub, uc) \neq (au, bu, cu).$$

Let $D_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}}(u,u,u) < 1$, then $D_{\mathcal{A}}(u) < 1$. Define $U := \log(e-u)$, then

$$e^{-U}ae^{U} \neq a, \qquad e^{-U}be^{U} \neq b \quad \text{and} \quad e^{-U}ce^{U} \neq c.$$

Now, define $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ as follows:

$$F(x,y,z) = e^{-\frac{x^2 z^2 U}{a^2 c^2}} x e^{\frac{y^2 z^2 U}{b^2 c^2}} \tag{3.13}$$

for every (x,y,z) in $\Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}}$. Clearly, F is a holomorphic function, $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i+j+k = 3$, $i,j,k = 0,1,2,3$, but $F(a,b,c) \neq a$, and similarly, we can show that there is a holomorphic map $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ with the required conditions such that $F(b,a,c) \neq b$ and $F(c,a,b) \neq c$. □

Example 3.4 Let $X = \mathbb{R}$ be the space of real numbers and let $F : X \times X \longrightarrow X$ be a function defined by $F(x,y,z) = x$ that satisfies the conditions of Theorem 3.1.

Example 3.5 Let X be a unital without of order complete semi-simple Banach algebra and let $F : X \times X \longrightarrow X$ be a function defined by $F(x,y,z) = e^{y^2z^2}xe^{-y^2z^2}$ that satisfies the conditions of Theorem 3.1. For example, let $X = M(G)$ be the measure space on a locally compact Hausdorff space G. Another algebra that we can choose is $\ell^1(G)$, where G is a locally compact discrete group.

Corollary 3.6 *Let $\mathcal{A}$ be a unital without of order semi-simple Banach algebra. For given $(a,b,c) \in \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \backslash Z(\mathcal{A} \times \mathcal{A} \times \mathcal{A})$, there is a holomorphic map $F : \Omega_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ satisfying the conditions $F(0,0,0) = 0$, $\frac{\partial F}{\partial x}(0,0,0) = \mathrm{id}_{\mathcal{A}}$, $\frac{\partial F}{\partial y}(0,0,0) = 0$, $\frac{\partial F}{\partial z}(0,0,0) = 0$, $\frac{\partial^2 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i + j + k = 2$, $i,j,k = 0,1,2$, and $\frac{\partial^3 F}{\partial x^i \partial y^j \partial z^k}(0,0,0) = 0$, where $i + j + k = 3$, $i,j,k = 0,1,2,3$, such that $F(a,b,c) \neq a$, $F(b,a,c) \neq b$ and $F(c,a,b) \neq c$.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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